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ON SOME GENERALIZATIONS OF A FAMOUS THEOREM

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The paper contains two generalizations of a well-known theorem. It says that if for a projectivity *p* on a line there exists a pair of distinct points a and b such that *p*(a)=b and *p*(b)=a, then the projectivity is an involution. This property can be generalized with respect to the dimension of the projective space.

1 Introduction

One of the most important theorems about projectivities on a line is the following ([1], p. 102): *If for a single point p which is not a double point of a projectivity π on the line we have the relations $\pi(p) = p'$ and $\pi(p') = p$, the projectivity is an involution.*

The above theorem can be formulated in another way:

If for a projective transformation f of a projective line l onto itself there exists a pair of distinct points p, q (one-dimensional simplex) such that f(p) = q, f(q) = p (i.e. the image under f of each vertex of the simplex lies in the opposite face hyperplane), then for any point $r \in l$ there exists a point s such that f(r) = s and f(s) = r.

We shall generalize the above taking into account the space dimension.

2 Main results

Notation:

P_n - n-dimensional projective space.

Let $A = \{a_1,..., a_m\}$ be a set of points. Then the symbol A_i denotes the set $A \setminus \{a_i\}$, i = 1,...,m.

By $J(a_1,..., a_k)$ we denote the join of points $a_1,..., a_k$.

Now we are ready to generalize the theorem quoted in the introduction.

Theorem 1.

If for a nonsingular collineation f in P_n there exists a set of points $A = \{a_0,..., a_n\}$ such that dimJ(A) = n and $f(a_i) \in J(A_i)$ i = 0,1,...,n, then for any set of points $C' = \{c_1,..., c_n\}$ there exists a point c_0 such that $\dim J(C_i,f(c_i)) \leq n-1$ for all i = 0,...,n, where $C = \{c_0,..., c_n\}$.

In other words, if for f there exists a simplex S such that its image under f is inscribed in S, then for arbitrarily chosen n vertices there exists an n+1-th vertex of a new simplex S' image of which is inscribed in S'.

Proof: If dimJ(C') < n-1, then we can take an arbitrary point of J(C') as c_0. Hence we may assume that dimJ(C') = n-1. Let an allowable coordinate system in be chosen in such a way that points a_i have the coordinates $a_{ij} = \delta_{ij}$ i,j = 0,1,...,n. Write f in the matrix form

$$y = Bx, \quad B = [b_{ij}]_{n+1\times n+1}$$

It follows from the assumption that $b_{ii} = 0$ for all i = 0,...,n. If now p(x) = $\sum_{i=0}^{n+1} p_i x^i$ is the characteristic polynomial of B, then $p_n = (-1)^n \sum_{i=0}^{n} b_{ii} = 0$.

Now assume that we are given linearly independent points $c_1,\ldots, c_n$. Let us consider such a new allowable coordinate system that the coordinates c_{ij} of points c_i are equal δ_{ij} i = 1,...,n, j = 0,...,n. We shall show the existence of a point $c_0 = (x_0,\ldots, x_n)$ such that dimJ(C_i,f(c_i)) ≤ n-1 for i = 0,...,n, and C = {$c_0,\ldots, c_n$}.

Denote by $D = [d_{ij}]_{n+1\times n+1}$ the matrix of f in the new coordinate system. Then $\sum_{i=0}^{n} d_{ii} = 0$, since p(x) is independent on the choice of a coordinate system. The points ($d_{0i},\ldots, d_{ni}$) are the images of c_i i = 1,...,n, and the point ($\sum_{i=0}^{n} d_{oi}x_i ,\ldots, \sum_{i=0}^{n} d_{ni}x_i$) is the image of c_0. Hence the relations dimJ({$c_0,\ldots, c_n$}\{c_i},f(c_i)) ≤ n-1 hold if and only if

$$d_{ii}x_0 - d_{0i}x_i = 0 \quad \text{for } i = 1,\ldots,n. \tag{1}$$

Similarly, the relation dimJ({$c_1,\ldots, c_n$,f(c_0)) ≤ n-1 holds if and only if

$$\sum_{i=0}^{n} d_{oi}x_i = 0 \tag{2}$$

The necessary and sufficient condition for the existence of numbers x_i not all zero satisfying (1) and (2) is the following:

$$\begin{vmatrix} d_{oo} & d_{o1} & \cdots & d_{on} \\ d_{11} & -d_{o1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ d_{nn} & 0 & \cdots & -d_{on} \end{vmatrix} = 0. \tag{3}$$

Add to the first row of the above determinant the all remaining rows. Since $\sum_{i=0}^{n} d_{ii} = 0$, the condition (3) holds.

Theorem 2.

If for a nonsingular collineation f in P_n there exists a set of points $A = \{a_0,..., a_n\}$ such that $\dim J(A) = n$ and $a_i \in J(f(A_i))$ $i = 0,1,...,n$, then for any set of points $C' = \{c_1,..., c_n\}$ there exists a point c_0 such that $\dim J(c_i,f(C_i)) \leq n-1$ for all $i = 0,...,n$, where $C = \{c_0,..., c_n\}$.

In other words, if for f there exists a simplex S such that S is inscribed in its image under f, then for arbitrarily chosen n vertices there exists an n+1-th vertex of a new simplex S' which is inscribed in its image under f.

Proof. Since $a_i \in J(f(A_i))$, $f^{-1}(a_i) \in J(A_i))$, $i = 0,1,...,n$. Hence there exists a point c_0 such that $\dim J(C_i,f^{-1}(c_i)) \leq n-1$ for all $i = 0,...,n$. Consequently, $\dim J(c_i,f(C_i)) \leq n-1$, all i.

A projective collineation f in P_n is called *normal cyclic* collination [2] if:

1) $f^{n+1} = id$; 2) there exists such a point a that points a, f(a),...,$f^n(a)$ are linearly independent.

Notice that this a natural generalization of the notion of an involution on a projective line. In fact, if we put n = 1 in the above, we obtain a projectivity of period 2, i.e. an involution.

Since the image of the simplex af(a)...$f^n(a)$ is inscribed in it, and, at the same time, the simplex is inscribed in its image, we obtain

Corollary.

For any normal cyclic collineation f in P_n and any set $A = \{a_1,..., a_n\}$ of points there exist points a_o and a^* such that $\dim J(A,f(a_o)) \leq n-1$, $\dim J(a_o,A\setminus\{a_i\},f(a_i)) \leq n-1$, $\dim J(a^*,f(A)) \leq n-1$ and $\dim J(f(a^*),f(A\setminus\{a_i\}),a_i) \leq n-1$.

3 Conclusions

The theorem generalized in the paper is one of the fundamental properties in projective geometry. We show that even in higher dimensions this property holds. It should be noticed that there are other generalizations of this property, e.g. [2], [3]. These generalizations concern the cyclicity of projective transformations, since an involution is a cyclic transformation of cycle 2. More precisely, if there exists a point p such that $f^k(p) = p$ and there are n+1 linearly independent points among points p, f(p),..., $f^k(p)$, then projective transformation f is exactly k-cyclic.

References:

[1] Veblen O., Young J.V.: Projective Geometry, Boston 1910.

[2] Witczyński K.: Projective Collineations as Products of Cyclic Collineations, Demonstr. Math. 12, 1979, pp. 1111-1125.

[3] Witczyński K.: Conditions goemetriques de cyclicite des collineations projectives de l’espace P^n, Demonstr. Math. 9, 1976, pp. 583-592.